Drawing Diagrams and Solving Word Problems: A Study of a Sample of Bruneian Primary and Secondary School Children

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Abstract:
Drawing diagrams is often recommended as a strategy for problem solving. The mathematics education literature suggests that teachers use diagrams as a problem solving strategy but provides little guidance on the kinds of diagrams teachers should draw and how teachers could teach pupils to generate their own diagrams. The findings from two separate studies, one with primary and the other with secondary school children, on the nature and use of diagrams in solving problems are described. The paper discusses the effectiveness of diagrams in helping children solve word problems, and implications for classroom instruction.

Introduction

Solving word problems in mathematics is a difficult topic to teach in both primary and secondary schools. Research into children's performance in public examinations in Brunei reveals that only about 1% of the pupils attempted any kind of diagram in solving word problems (Lopez-Real, Veloo & Maawiah, 1992). Although drawing of diagrams as a problem solving strategy has long been advocated by mathematics educators, (Polya,1957; Schoenfeld, 1985; Hyde.A.A. et al. 1991) empirical evidence on its effectiveness has been scarce. Children have been found to perform better in mathematical problem solving situations when diagrams are used by teachers to elicit appropriate mental images (Yancey,V. et al,1989; Driscoll, 1979; Riedesel 1969).

However, little has been written about how pupils should be taught to generate their own diagrams (Moses, 1982; Nelson 1983). The extent to which children use this strategy in problem solving, or how spontaneously they use this strategy has not been fully investigated. This paper discusses the results from two studies in primary and secondary schools in Brunei Darussalam on children's use of diagrams in solving word problems.

Methodology

The sample in the first study (Lopez-Real & Veloo,1993) consisted of 96 children from two primary schools in Bandar Seri Begawan, the capital of Brunei Darussalam. Each child was given a set of ten word problems to solve. The Newman interview procedure which classifies students' errors on written mathematical tasks as either *reading, comprehension, transformation, process skills* or *encoding* (Newman, 1977) was used to study unsuccessful instances. Errors that were identified as *comprehension* or *transformation* were followed up with the following procedure. First, the pupil was asked to draw a diagram of the problem and to include any important information from the problem into the diagram and then to try to solve the problem again. No help or assistance was given by the interviewer. If the pupil was unable to draw a diagram or was still unsuccessful after having drawn one, he or she was then presented with a diagram drawn by the authors and asked to try again using this diagram. The results are as follows.

Table 1: Successful and unsuccessful Cases after Drawing Diagrams

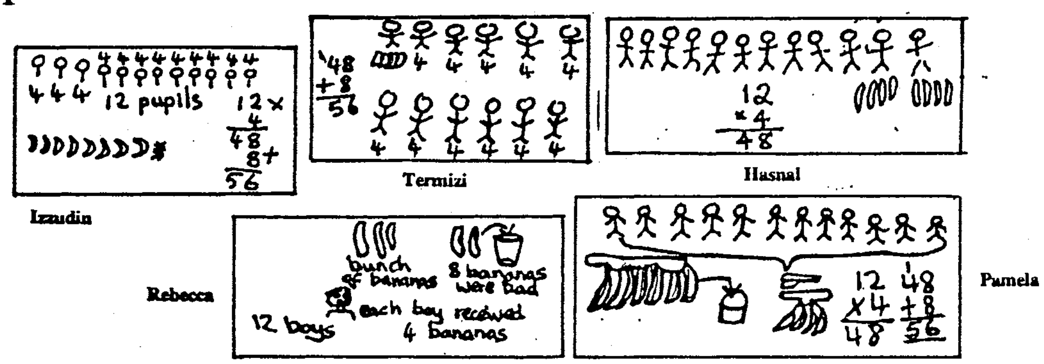
Able to draw Diagram		If Diagram Drawn		If Authors' Diagram Used	
Yes	No	Successful	Unsuccessful	Successful	Unsuccessful
107	19	41	66	52	26

The striking feature in Table 1 is that 38% (or 41 out of 107) of the cases in which pupils had previously failed to obtain a correct solution, they were now successful with no further assistance or help other than the suggestion to represent the problem pictorially. The responses to several of the problems used in the study are now discussed in more detail.

Discussion of selected diagrams

Problem 1: *A teacher brought a bunch of bananas to be shared equally among 12 boys. 8 of the bananas were bad and were thrown away. After this each boy received 4 bananas. How many bananas were in the bunch at the start?*

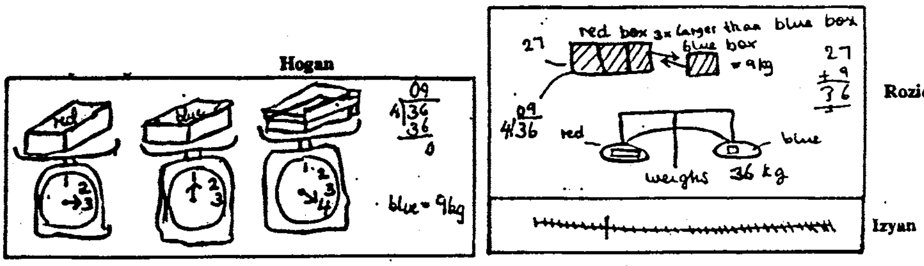
Figure 1



The diagrams in Fig.1 show that once a matching has been established between boys and bananas, the pupils were able to determine with ease the critical step 12 x 4. Izzudin's example is the most common type. Termizi shows a slight variation. Hasnal drew 4 bananas but did not proceed beyond the second pupil; this was sufficient to enable him to establish the matching between pupils and bananas. Pamela's diagram even shows a waste bin for the bad bananas but the representation of the twelve pupils enabled her to obtain 4x12. Rebecca's diagram shows a literal representation of the major elements of the problem but no matching of boys and bananas is evident.

***Problem 2.** A red box weighs 3 times as much as a blue box. The two boxes together weigh 36 kg. What is the weight of the blue box.*

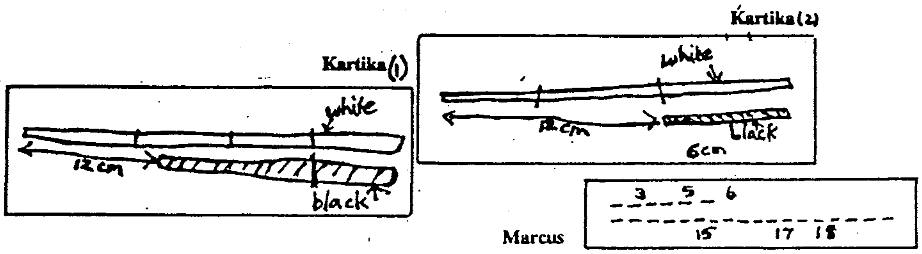
Figure 2



The most common diagram observed for this question is similar to that drawn by Rozie. The diagram expresses clearly the relationship between the red box and the blue box. A *concrete representation* of the problem is given in the diagram drawn by Hogan. He drew three balances the first with the pointer at 3, the second has the pointer at 1 and the third has the pointer at 4. He then successfully wrote down the step 36÷4. Izyan's diagram shows a line, marked out with 36 parts to represent the total weight 36kg. He counted along the line and marked a longer division on 9 and said that the weight of the blue box is 9kg. He then proceeded to calculate the weight of the red box as 36–9.

***Problem 7.** A white stick is 3 times longer than a black stick. The difference in their lengths is 12cm. What is the length of the black stick?*

Figure 3



Kartika made two attempts at drawing without any prompting from the interviewer. Having completed her first drawing, she immediately shook her head and said "No, that's not right". She proceeded to draw a second diagram and then wrote down the answer 6cm without hesitation. Marcus used a totally different strategy. His representation was in terms of "match sticks" and he used a trial-and-error approach.

Secondary school study

The secondary school study was a follow-up study to the primary school study described above. In particular, we wanted to investigate (a) the extent to which diagrams drawn by pupils at different levels of secondary schooling were *concrete, semi-concrete or symbolic* representations of the given problems, and (b) whether the differences in the types of representation were related to levels of schooling. Two hundred pupils from secondary 1 to secondary 4 of two secondary schools in Bandar Seri Begawan were each given a set of ten word problems to solve. These problems did not involve any knowledge of mathematical concepts or techniques beyond the secondary 1 level. The pupils were instructed to read each problem carefully and then to draw a diagram that they felt would help them solve the problem. They were also instructed to include all important information in their diagrams. Despite the instruction which was verbally emphasised when conducting the test, to draw diagrams for each problem, the distribution of cases for which there was no diagram varied from 5% to 64.5%. The number of children who actually drew diagrams against those who did not for each question is given in Table 2 below.

Table 2 : Distribution of cases with and without diagrams

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Diagram drawn	165	171	180	190	96	161	134	71	107	115
No diagram	35	29	20	10	104	39	66	129	93	85

Table 3 shows the success rate for each question.

Table 3: Distribution of cases with correct and incorrect solutions

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Correct	126	186	151	154	21	181	83	36	85	16
Wrong	74	14	49	46	179	19	117	164	115	184

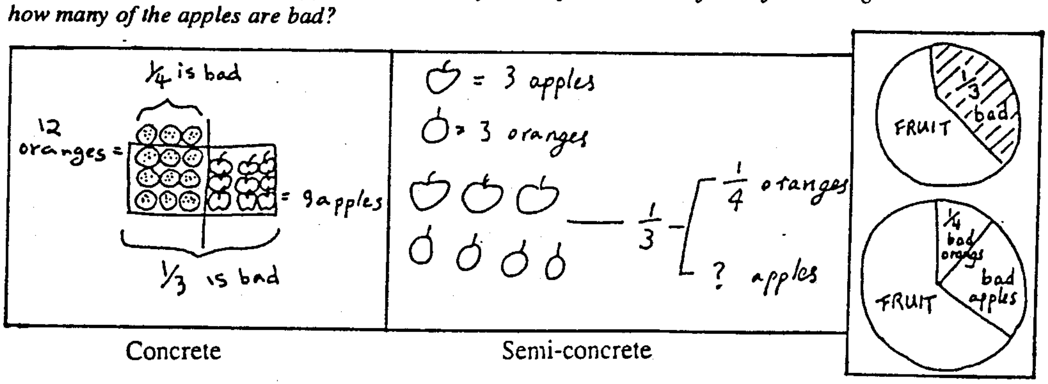
With regard to the type of representation, a diagram is classified as *Concrete* when students attempt to give a physical or literal representation of the objects in the problem. At the other end of the scale, where the representation is purely symbolic, it is classified as *Symbolic*. However, where the representation appears to lie somewhere between the two extremes it is categorised as *Semi-Concrete*. Examples of these are shown below.

Examples of diagrams drawn by secondary school pupils

The following are selected examples of diagrams drawn by secondary school pupils.

Problem 3: *A box contains 12 oranges and 9 apples. One-third of all the fruits is bad. If 1/4 of the oranges are bad, how many of the apples are bad?*

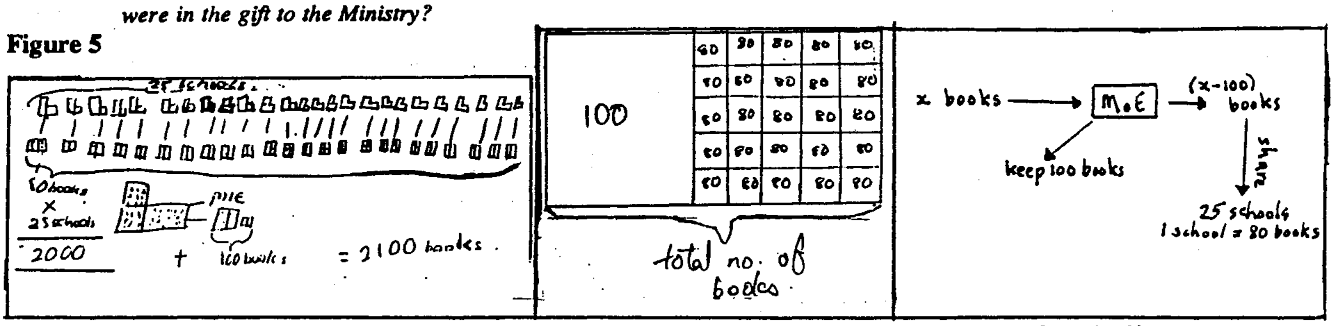
Figure 4



Concrete | Semi-concrete | Symbolic

***Problem 6:** A large number of books were received by the Ministry of Education as a gift. The Ministry kept back 100 books for its own library and the rest were shared equally among 25 schools. If each school received 80 books, how many books were in the gift to the Ministry?*

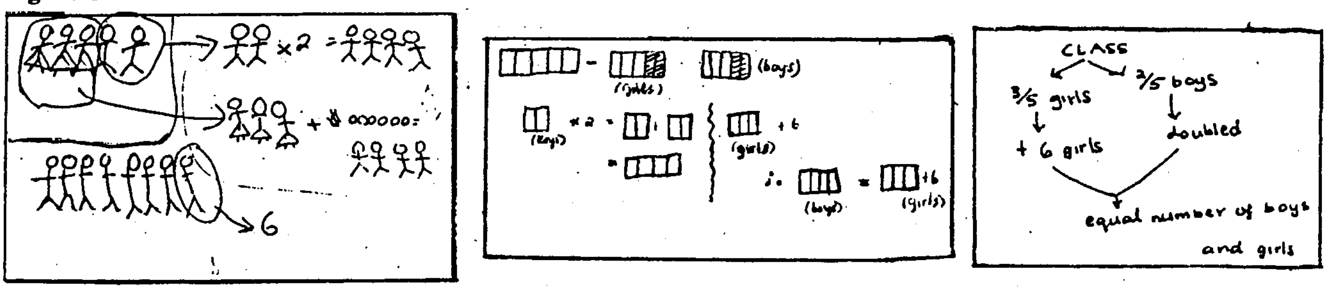
Figure 5



Concrete Semi-concrete Symbolic

***Problem 8:** In a certain class 3/5 of the pupils are girls. If the number of boys is doubled and 6 more girls join the class, there would be an equal number of boys and girls. How many pupils were in the class at the start?*

Figure 6



Concrete Semi-concrete Symbolic

Table 4 below shows the breakdown of results for each question with the diagrams classified under the categories *Concrete, Semi-concrete or Symbolic.*

Table 4: Distribution of Concrete, Semi-concrete and Symbolic diagrams

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
No diagram	35	29	20	10	104	39	66	129	93	85
Concrete	72	63	69	6	60	33	29	14	51	44
Semi-concrete	67	75	76	23	19	55	60	20	22	44
Symbolic	26	33	35	161	17	73	45	37	34	27

In this study no interviews were conducted and so we can only speculate on the reasons for children not having drawn a diagram. The fact that the results in Table 2 vary enormously from question to question suggests that the nature of the problem itself is an important factor. A geometric problem will readily lend itself to a visual representation and it is therefore no surprise that this question (Table 4) has the highest frequency for diagrams drawn. This was the only problem that referred to a geometric figure. Omitting this case, two extreme cases, are *Problem 3* with 90% drawn and *Problem 8* with only 35.5% drawn. Both problems involve fractions and neither involve large numbers, so why the difference? It can be argued that if pupils immediately recognise how to tackle a problem and feel confident about solving it then they will consider the instruction to draw a diagram quite pointless and simply ignore it. This is a reasonable argument, but the results in Table 3 showing the success rate for each question, certainly belie this argument. Comparing *success* in solving the problem and *diagrams drawn* for each question we find that the proportion of pupils who successfully solved a problem was higher among those who drew diagrams than those who did not.

Table 4 shows the breakdown of results for each question with the diagrams classified under three categories described earlier. The nature and difficulty of the problem has a strong bearing on the type of diagram drawn by pupils. Our analysis shows pupils in Secondary 1 and 2 tend to draw diagrams that are more of the concrete type than pupils in Secondary 3 and 4. In order to determine whether there was a change in the type of diagrams drawn by pupils at different levels, a χ^2 test was used to analyse the responses for each question. Contingency tables involving levels of schooling (Secondary 1&2 and 3&4) against type of diagram drawn were obtained for all the 10 questions. Table 5 gives a summary of the χ^2 analysis.

Table 5: Distribution of the level of significance for each question

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Significance at $p<0.005$	No	Yes	Yes	No	No	Yes	Yes	Yes	Yes	Yes

In most cases where there was a significant difference, the direction of change was that the Secondary 3 and 4 pupils drew more diagrams at the symbolic end of the continuum and less at the concrete end than the Forms 1 and 2 pupils.

Discussion of results

Cues and mis-cues

One of the most common behaviours exhibited by primary school children *prior* to being asked to draw a diagram was to perform an operation initiated by a *cue* word in the problem. The word "times" in *Problems 2* and *7* often resulted in operations like 36x3 and 12x3. Stuart's comment while trying *Problem 1* highlights this type of behaviour: "It says *thrown away* so it's got to be a *subtract* somewhere!" Such cue words act as mis-cues almost as frequently as they cue the appropriate operation. We suspect that the strategy of looking for cue words is actively encouraged by many textbooks and teachers. The speed with which children launched into operating on the numbers (without apparent reference to their meaning) was very striking. Typically, 12x8 and 12-8 were frequent attempts at *Problem 1*, while 36÷3 and 12÷3 were even more common for *Problems 2* and *7* respectively.

Function of Diagrams.

Given that nearly 38% of the primary school pupils interviewed were successful in solving the problems after simply drawing a diagram, can we identify what function the diagram is playing in these cases? Two elements are involved in comprehending (or understanding) a problem. The first arises from the Newman instruction "Tell me what the question is asking you to do." This is effectively asking whether the child understands the *structure* of the problem. It does not probe any understanding of the second element which involves understanding of the *concepts.*It was clear that some children understood the "sticks" problem in terms of knowing that they had to find the length of the shorter stick but did not have an understanding of the "difference" concept. From our observations, it appears that the very act of drawing a diagram focuses the child's attention on what the numbers *represent* rather than the numbers as abstract entities. In this sense, the function of a diagram may be to act as an alternative form of "expressing the problem in one's own words". Having focused on the meaning of the numbers and their relationships within the problem, it then appears that the diagram can act as a key aid in the *transformation* stage.

Helpful Features of Diagrams

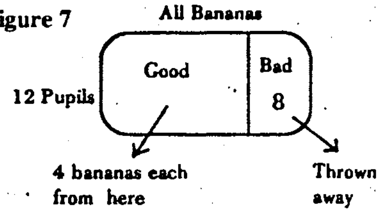
Two elements featured regularly in diagrams that led to successful outcomes. First, the representation of *relationships* in a clear, visual form and second, the *incorporation of numerical information* from the problem into the diagram itself. Irrespective of whether the diagram drawn was *concrete, semi-concrete* or *symbolic*, it was clearly evident that the feature in diagrams that led to successful outcomes was the representation of relationships involved.

Stages in diagram development

A sizable proportion of the pupils were unable to succeed even with the authors' diagrams in the primary school study. The authors' diagrams were all at the symbolic end of the continuum described earlier. (The author's diagram for *Problem 1* is shown below.) It could well be the case that some children were not ready for such representations and yet might have been able to interpret a more physical or literal version. This leads us to conjecture that children may need to progress through stages of development in the drawing and interpretation of diagrams for problem-solving.

They need to move from a concrete form of representation to a more symbolic form.

Figure 7



Conclusion

Most children in the primary school study showed a general willingness to use the strategy of drawing diagrams. There are strong indications in the first study that drawing diagrams can be an important strategy in problem solving but one that is not exploited in the classroom. The desire to push children too quickly into abstract manipulation and algorithms may be one reason for this. And possibly teachers may have the perception (which is passed on to the children) that drawing diagrams is somehow less mathematically respectable than algebraic manipulation.

In our investigation involving secondary school pupils we have found that the type of diagrams drawn by children increases in sophistication as they mature. This has implications for teachers intending to use diagrams as a problem solving strategy. Young children need to be initiated into the technique of drawing diagrams beginning with a concrete representation of the problem and then moving to semi-concrete and later into more abstract forms of representations as they become more mature.

The results of the two studies strongly suggest that diagrams drawn by pupils which represent relational aspects of the problem provide a definite aid to a successful solution of a problem.

In these two studies we have found that there is a lack of spontaneity, often bordering on resistance on the part of the pupils, in drawing diagrams to solve problems. There are strong indications that children both in primary and secondary schools are not introduced to this technique of problem solving, which is a matter of concern. We have shown here that drawing diagrams to solve problems is a useful strategy. However, if this strategy is to be taken seriously it must be taught. A rich diversity of examples and techniques should be introduced to pupils. We feel that there is a rich field here for further research both in identifying the salient features of helpful diagrams and in developing suitable curriculum materials.

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